# SOLUTIONS TO SELECTED QUESTIONS IN HOMEWORK 14

MATH 241

15.3.4

*Proof.*

$$A(\alpha) = \int_0^\pi \sin x \cos \alpha x dx = \frac{1}{2}\int_0^\pi \sin(1+\alpha)x + \sin(1-\alpha)x dx = \frac{1}{2}[-\frac{\cos(1+\alpha)x}{1+\alpha} - \frac{\cos(1-\alpha)x}{1-\alpha}]|_0^\pi$$
$$= \frac{1}{2}[\frac{1-\cos(1+\alpha)\pi}{1+\alpha} + \frac{1-\cos(1-\alpha)\pi}{1-\alpha}]$$

$$B(\alpha) = \int_0^\pi \sin x \sin \alpha x dx = \frac{1}{2}\int_0^\pi \cos(1-\alpha)x - \cos(1+\alpha)x dx = \frac{1}{2}[\frac{\sin(1-\alpha)x}{1-\alpha} - \frac{\sin(1+\alpha)x}{1+\alpha}]|_0^\pi$$
$$= \frac{1}{2}[\frac{\sin(1-\alpha)\pi}{1-\alpha} - \frac{\sin(1+\alpha)\pi}{1+\alpha}]$$

Therefore

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{1}{2}[\frac{1-\cos(1+\alpha)\pi}{1+\alpha} + \frac{1-\cos(1-\alpha)\pi}{1-\alpha}]\cos \alpha x + \frac{1}{2}[\frac{\sin(1-\alpha)\pi}{1-\alpha} - \frac{\sin(1+\alpha)\pi}{1+\alpha}]\sin \alpha x d\alpha$$

□

15.3.17

*Proof.* For an $f$ defined on $(0,\infty)$, then the integral in the problem is the coefficient of Fourier cosine integral, so $A(\alpha) = e^{-\alpha}$, therefore the Fourier integral becomes

$$f(x) = \frac{2}{\pi}\int_0^\infty e^{-\alpha}\cos \alpha x d\alpha = \frac{2}{\pi}\int_0^{-\infty} e^{\alpha}\cos \alpha x(-d\alpha) = \frac{2}{\pi}\int_{-\infty}^0 e^{\alpha}\cos \alpha x d\alpha$$

Then by the formula

$$\int e^t \cos \lambda t dt = \frac{e^t(\lambda \sin \lambda t + \cos \lambda t)}{\lambda^2+1}$$

In this situation $\lambda = x, t = \alpha$, so we get

$$\frac{2}{\pi}\frac{e^{\alpha}(x\sin \alpha x + \cos \alpha x)}{x^2+1}|_{-\infty}^0$$

**Note** that here we are evaluating with respect to $\alpha$, not $x$! When $\alpha \to -\infty$, $e^\alpha \to 0$ but $(x\sin \alpha x + \cos \alpha x)$ is bounded (again, notice that $x$ is not changing! That's why it is fixed, otherwise if $x$ is allowed to go to infinity, this may no longer be true), so the evaluation at $-\infty$ is zero. Therefore the answer is $f(x) = \frac{2}{\pi}\frac{1}{x^2+1}$. □

Fall 10, #11

*Proof.*

$$\mathcal{F}\{f\}(\alpha) = \int_{-1}^{1} 2e^{i\alpha x}dx = \frac{2}{i\alpha}e^{i\alpha x}|_{-1}^{1} = \frac{2(e^{i\alpha} - e^{-i\alpha})}{i\alpha} = \frac{4i\sin\alpha}{i\alpha} = \frac{4\sin\alpha}{\alpha}$$

$g'' = f$ is routine to check. Therefore

$$\mathcal{F}\{f\}(\alpha) = \mathcal{F}\{g''\}(\alpha) = (-i\alpha)^2\mathcal{F}\{g\}(\alpha)$$

Therefore $\mathcal{F}\{g\}(\alpha) = -\frac{1}{\alpha^2}\mathcal{F}\{f\}(\alpha) = -\frac{4\sin\alpha}{\alpha^3}$. □